25-3-2008

Applied Signal Processing – 17/4/2007 9.15-12.15h **VERSION B**

1. Find the DTFT of the two-sided sequence $x[n]=(\frac{1}{3})^{|n|}$. **(1)**

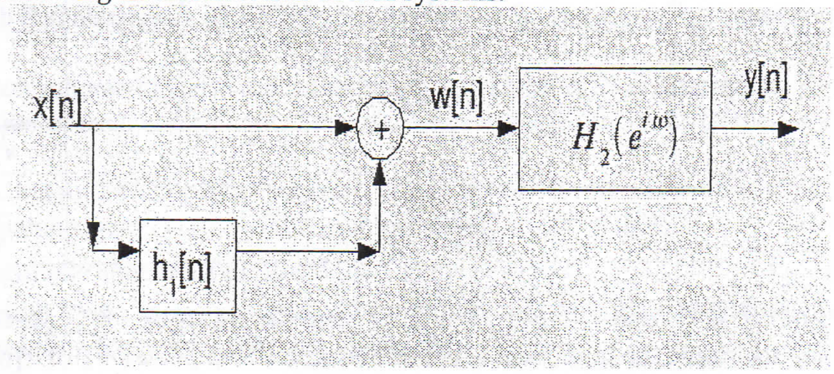
2. Find the z-transform of the following sequences. Wherever convenient, use the properties of the z-transform to make the solution easier:

(a) $x[n]=(\frac{1}{3})^{n}\mu[-n]$ **(0.5)**

(b) $x[n]=(\frac{1}{2})^{n}\mu[-n+2]+3^{n}\mu[n-1]$ **(0.5)**

(c) $x[n]=2n(\frac{1}{2})^{n}\mu[n+1]$ **(0.5)**

3. Consider the following interconnection of LTI systems:



where $h_1[n]=\delta[n-1]$ and

$$H_2(e^{i\omega}) = \begin{cases} 1 \ for \ |\omega| \leq \frac{\pi}{2} \\ 0 \ for \ \frac{\pi}{2} \leq |\omega| \leq \pi \end{cases}$$

Find the frequency response and the unit impulse response of the system. **(1+1)**

4. Consider the linear constant-coefficient difference equation

$$y[n]-\frac{1}{4}y[n-1]-\frac{1}{8}y[n-2]=2x[n-1]$$

Determine y[n] for $n \geq 0$ when $x[n]=\delta[n]$ and $y[n]=0, n<0$ **(1.5)**

5. As part of the first stage in a radix 2 FFT, a sequence x[n] of length N=8 is decomposed into 2 sequences of length 4 as:

$$f_0[n]=x[2n], n=0,1,2,3$$
$$f_1[n]=x[2n+1], n=0,1,2,3$$

We compute a 4 point DFT of each of these 2 sequences as

$$f_0[n] \xleftrightarrow[4]{\text{DFT}} F_0[k] \qquad f_1[n] \xleftrightarrow[4]{\text{DFT}} F_1[k]$$

The specific values of $F_0[k]$ and $F_1[k]$ (k=0,1,2,3), obtained from the length N=8 sequence in question are listed in the table below:

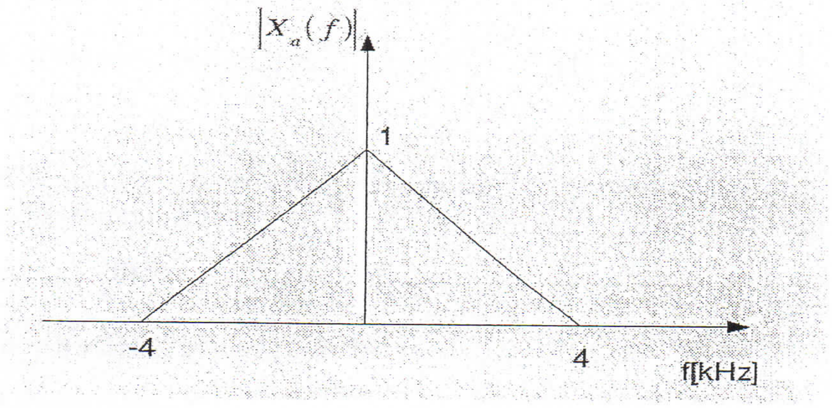
k	**0**	**1**	**2**	**3**
$F_0[k]$	0	0	1	1
$F_1[k]$	0	0	-i	$\frac{1}{\sqrt{(2)}}(-1+i)$
W_8^k	1	$\frac{1}{\sqrt{(2)}}(1-i)$	-i	$\frac{1}{\sqrt{(2)}}(1+i)$

(a) From the values of $F_0[k]$ and $F_1[k]$ (k=0,1,2,3) and the values of W_8^k (k=0,1,2,3), provided in the table, determine the numerical value of the actual N=8 point DFT of x[n] denoted $X_s[k]$. That is, determine the numerical value of $X_s[k]$ for k=0,1,...,7. **(1.5)**

(b) The underlying length N=8 sequence x[n] may be expressed as

$x[n]=\frac{1}{4}e^{2\pi i\frac{k_1}{8}}+\frac{1}{4}e^{2\pi i\frac{k_2}{8}}$ where k_1 and k_2 are both integers between 0 and 7. Given the values of $X_s[k]$ determined in part (a) determine the numerical values of k_1 and k_2 . **(0.5)**

6. Let $x_1(t)$ be an analog signal, with an amplitude spectrum $|X_a(f)|$ shown in the following figure:



A discrete-time signal $x_1(n)$ was generated by sampling the signal $x_a(t)$ using the sampling frequency $F_{s1}=8\,kHz$.

We wish to design a digital system that reduces the sampling frequency of the signal $x_1(n)$ to $F_{s2}=6\,kHz$ such that aliasing does not appear. Let $x_2(m)$ be the resulting output signal.

a) Sketch the block scheme of the system (including up-samplers, down-samplers, filters, etc.) and explain the function of each component. State the necessary specifications of the components of the system. **(1)**
b) Will the reduction in the sampling frequency cause a loss of information, i.e. is it possible to reconstruct the analog signal $x_a(t)$ from the output signal $x_2(m)$? **(0.5)**
c) Sketch the magnitude spectra of all the signals in the system (i.e. $|X_1(f)|$, $|X_2(f)|$ and possible intermediate signals) for frequencies up to at least 2x the sampling frequency. State the value of the corresponding sampling frequency for each graph. **(0.5)**

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(1)

$$X(e^{i\omega}) = \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n e^{-i\omega n} + \sum_{n=-\infty}^{0}\left(\frac{1}{3}\right)^{-n} e^{-i\omega n} - 1$$

$$= \frac{1}{1-\frac{1}{3}e^{-i\omega}} + \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n e^{i\omega n} - 1$$

$$= \frac{1}{1-\frac{1}{3}e^{-i\omega}} + \frac{1}{1-\frac{1}{3}e^{i\omega}} - 1 = \frac{8/9}{10/9 - \frac{2}{3}\cos\omega}.$$

(2) (a) $x[n] = \left(\frac{1}{3}\right)^n \mu[-n]$

$$X(z) = \sum_{n=-\infty}^{0}\left(\frac{1}{3}\right)^n z^{-n} = \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{-n} z^n = \frac{1}{1-3z}$$

(b) $x[n] = \left(\frac{1}{2}\right)^n \mu[-n+2] + 3^n \mu[n-1]$

$$X(z) = \sum_{n=-\infty}^{2}\left(\frac{1}{2}\right)^n z^{-n} + \sum_{n=1}^{\infty} 3^n z^{-n}$$

$$= \sum_{n=-2}^{\infty}\left(\frac{1}{2}\right)^{-n} z^n + \sum_{n=1}^{\infty} 3^n z^{-n}$$

$$= \left(\frac{1}{2}\right)^{+2} z^{-2}\left(\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{-n} z^n\right) + 3z^{-1}\sum_{n=0}^{\infty} 3^n z^{-n}$$

$$= \frac{1}{4z^2}\cdot\frac{1}{1-2z} + \frac{3}{z}\cdot\frac{1}{1-\frac{3}{z}} =$$

$$= \frac{1}{4z^2(1-2z)} + \frac{3}{z-3}$$ (can be simplified more).

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2 (c) $x[n] = 2n\left(\frac{1}{2}\right)^n \mu[n+1]$

Use here: $Z(n\,g[n]) = -z\,\dfrac{dG(z)}{dz}$

Let $g[n] = 2\cdot\left(\frac{1}{2}\right)^n \mu[n+1]$.

$$G(z) = 2\sum_{n=-1}^{\infty}\left(\frac{1}{2}\right)^n z^{-n} = 2\cdot\left(\frac{1}{2}\right)^{-1} z\left(\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n}\right)$$

$$= 4z\,\frac{1}{1-\frac{1}{2z}} = \frac{8z^2}{2z-1}.$$

$$\frac{dG(z)}{dz} = \frac{16z\cdot(2z-1) - 8z^2\cdot 2}{(2z-1)^2} = \frac{32z^2-16z-16z^2}{(2z-1)^2}$$

$$= \frac{16(z^2-z)}{(2z-1)^2}$$

So $Z(x[n]) = -z\left(\dfrac{16(z^2-z)}{(2z-1)^2}\right) = \dfrac{-16z^3+16z^2}{(2z-1)^2}.$

③

To find the unit sample response, let $x[n] = \delta[n]$.

The output of the adder is

$$w[n] = \delta[n] + \delta[n-1]$$

So

$$y[n] = h_2[n] + h_2[n-1]$$

Here

$$h_2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_2(e^{i\omega})e^{ni\omega}\,d\omega = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{in\omega}\,d\omega$$

$$= \frac{\sin(n\pi/2)}{n\pi}$$

So, the unit impulse response of the whole system is:

$$h[n] = \frac{\sin(n\pi/2)}{n\pi} + \frac{\sin((n-1)\pi/2)}{(n-1)\pi}.$$

For the frequency response:

$$W(e^{i\omega}) = 1 + e^{-i\omega}$$

So

$$H(e^{i\omega}) = W(e^{i\omega})H(e^{i\omega}) = (1 - e^{-i\omega})H_2(e^{i\omega})$$

$$= \begin{cases} 1 + e^{-i\omega} & (|\omega| \leq \pi/2) \\ 0 & (\pi/2 < |\omega| \leq \pi) \end{cases}$$

(4)

$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-2] = 2x[n-1]$$

When $x[n] = \delta[n]$, $y[n] = h[n]$ (impulse response)

We calculate $H(e^{i\omega})$ from:

$$Y(e^{i\omega}) - \frac{1}{4}Y(e^{i\omega})e^{-i\omega} - \frac{1}{8}Y(e^{i\omega})e^{-2i\omega} = 2X(e^{i\omega})e^{-i\omega}$$

So $$H(e^{i\omega}) = Y(e^{i\omega}) / X(e^{i\omega}) = \frac{2e^{i\omega}}{1 - \frac{1}{4}e^{-i\omega} - \frac{1}{8}e^{-2i\omega}}$$

$$= \frac{A}{1 - \frac{1}{2}e^{-i\omega}} + \frac{B}{1 + \frac{1}{4}e^{-i\omega}} = \frac{A(1 + \frac{1}{4}e^{i\omega}) + B(1 - \frac{1}{2}e^{-i\omega})}{(1 - \frac{1}{2}e^{-i\omega})(1 + \frac{1}{4}e^{-i\omega})}$$

In that case
$$\begin{cases} A + B = 0 \\ +\frac{1}{4}A - \frac{1}{2}B = 2 \end{cases} \quad \text{or} \quad \begin{cases} A + B = 0 \\ -\frac{1}{4}B - \frac{1}{2}B = 2 \end{cases}$$

$$\left(B = -\frac{8}{3}, A = \frac{8}{3}\right)$$

$$= \frac{8/3}{1 - \frac{1}{2}e^{-i\omega}} - \frac{8/3}{1 + \frac{1}{4}e^{-i\omega}}$$

So $$h[n] = \frac{8}{3}\left(\frac{1}{2}\right)^n \mu[n] - \frac{8}{3}\left(\frac{1}{4}\right)^n \mu[n] = y[n]$$

(5) Here we use (see derivation of FFT):

$$X_s[k] = F_0[k] + W_8^k F_1[k] \quad (k=0,1,2,3)$$

$$X_s[k+4] = F_0[k] - W_8^k F_1[k]$$

This gives:

$X_s[0] = 0$

$X_s[4] = 0$

$X_s[1] = 0$

$X_s[5] = 0$

$X_s[2] = 1 + (-i)(-i) = 0$

$X_s[6] = 2$

$X_s[3] = 2$

$X_s[7] = 0$

So $X_s[k] = 2\delta[k-3] + 2\delta[k-6]$.

Since we know that the IDFT of $\delta(k-k_0) = \frac{1}{N} W_N^{-k_0}$

we have:

$$x[n] = \frac{1}{4} e^{2\pi i/8 \cdot 3} + \frac{1}{4} e^{2\pi i/8 \cdot 6}$$

So $k_1 = 3$, $k_2 = 6$.

(6)

$$x_1[n] \longrightarrow \boxed{\uparrow I} \xrightarrow{v[\ell]} \boxed{h[\ell]} \xrightarrow{w[\ell]} \boxed{\downarrow D} \longrightarrow x_2[m]$$

(Block diagram)

(a)

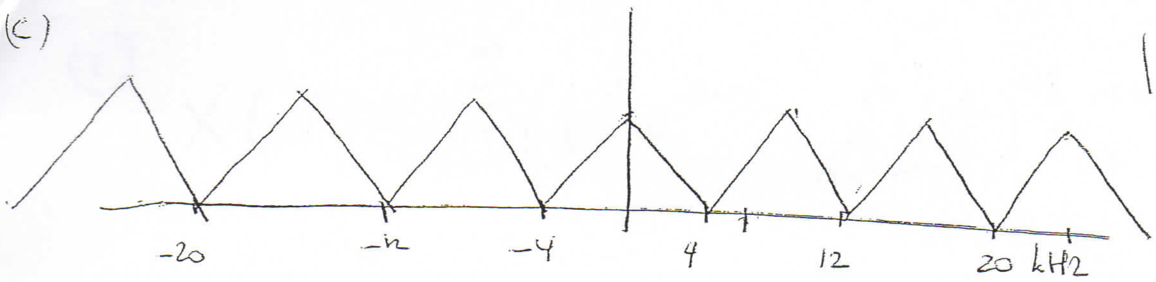
$\boxed{\uparrow I}$ Interpolator, inserts $I-1$ zero-samples between the samples of $x_1[n]$. Here the sampling frequency should go up by a factor 3 $(=I)$

$\boxed{h[\ell]}$ Digital low-pass filter, that removes all ~~flux above~~ frequencies above $F_c = 3$ kHz. Operates at sampling frequency 24 kHz.

$\boxed{\downarrow D}$ Decimator, retaining only each D^{th} sample. In this case $D=4$.

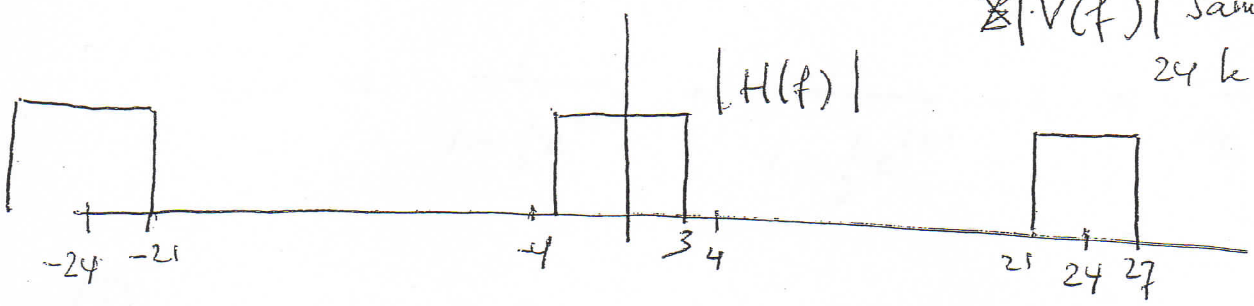
(b) Since the bandwidth of the signal is larger than $F_{s2} = 6$ kHz, it is not possible to reconstruct $x_a(t)$ from $x_2[m]$. So, some information is lost.

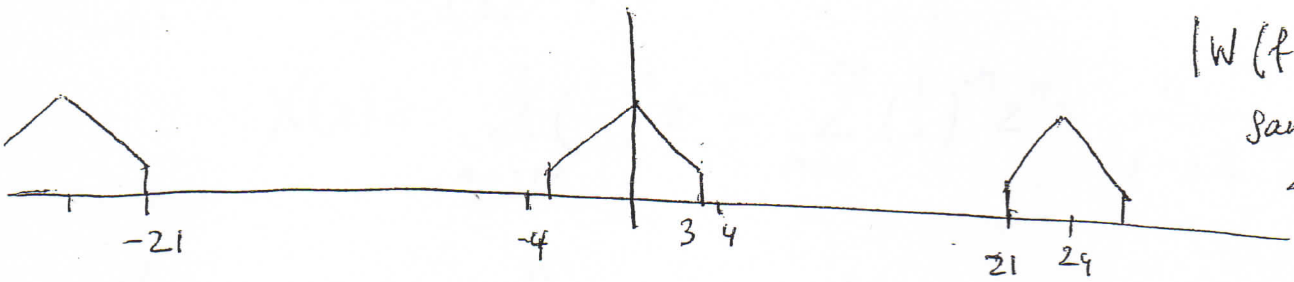
(c)



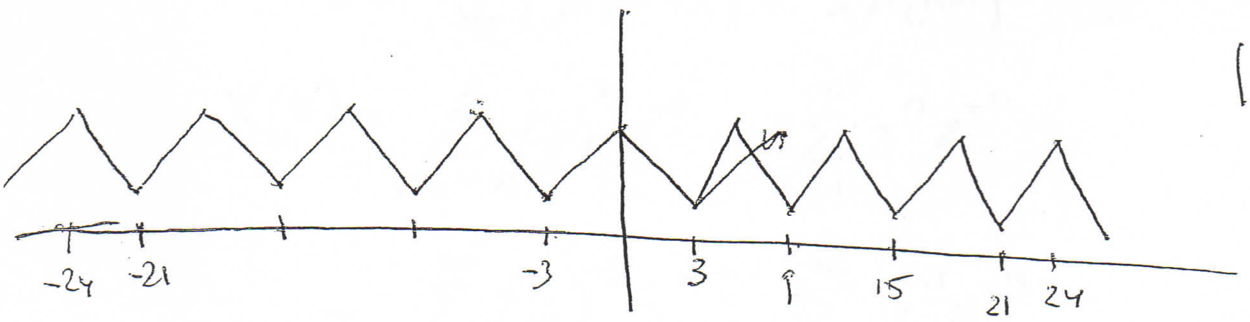
$|X_1(f)|$ Sampled at 8 kHz

Same for ~~Z~~ $|V(f)|$ sampled at 24 kHz.





$|W(f)|$ sampled at 24 kHz



$|X_2(f)|$ Sampled at 6 kHz